

Entanglement entropy in the Lipkin-Meshkov-Glick model

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We analyze the entanglement entropy in the Lipkin-Meshkov-Glick model, which describes mutually interacting spins half embedded in a magnetic field. This entropy displays a singularity at the critical point that we study as a function of the interaction anisotropy, the magnetic field, and the system size. Results emerging from our analysis are surprisingly similar to those found for the one-dimensional XY chain.

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Within the last few years, entanglement properties of spin systems have attracted much attention. As initially shown in one-dimensional (1D) spin chains [1, 2, 3, 4], observables measuring this genuine quantum mechanical feature are strongly affected by the existence of a quantum phase transition. For instance, the so-called concurrence [5] that quantifies the two-spin entanglement displays some nontrivial universal scaling properties. Similarly, the von Neumann entropy which rather characterizes the bipartite entanglement between any two subsystems scales logarithmically with the typical size L of these subsystems at the quantum critical point, with a prefactor given by the central charge of the corresponding theory [3, 4, 6, 7]. Note that the role played by the boundaries in these conformal invariant systems has been only recently elucidated [8]. Apart from 1D systems, very few models have been studied so far [9, 10, 11, 12, 13], either due to the absence of exact solution or to a difficult numerical treatment. In this context, the Lipkin-Meshkov-Glick (LMG) model [14, 15, 16] discussed here has drawn much attention since it allows for very efficient numerical treatment as well as analytical calculations. Introduced by Lipkin, Meshkov and Glick in Nuclear Physics, this model has been the subject of intensive studies during the last two decades because of its relevance for quantum tunneling of bosons between two levels. It is thus of prime interest to describe in particular the Josephson effect in two-mode Bose-Einstein condensates [17, 18]. The entanglement properties of this model have been already discussed through the concurrence, which exhibits a cusp-like behavior at the critical point [19, 20, 21, 22] as well as interesting dynamical properties [23]. Note that similar results have also been obtained in the Dicke model [24, 25] which can be mapped onto the LMG model in some cases [26], or in the reduced BCS model [27].

In this letter, we analyze the von Neumann entropy computed from the ground state of the LMG model. We show that, at the critical point, it behaves logarithmically with the size of the blocks L used in the bipartite decomposition of the density matrix with a prefactor that

depends on the anisotropy parameter tuning the universality class. We also discuss the dependence of the entropy with the magnetic field and stress the analogy with 1D systems.

The LMG model is defined by the Hamiltonian

$$H = -\frac{\lambda}{N} \sum_{i < j} (\sigma_x^i \sigma_x^j + \gamma \sigma_y^i \sigma_y^j) - h \sum_i \sigma_z^i, \quad (1)$$

where σ_α^k is the Pauli matrix at position k in the direction α , and N the total number of spins. This Hamiltonian describes a set of spins half located at the vertices of a N -dimensional simplex (complete graph) interacting via a ferromagnetic coupling ($\lambda > 0$) in the xy -spin plane, γ being an anisotropy parameter and h an external magnetic field applied along the z direction. The Hamiltonian (1) can be written in terms of the total spin operators $S_\alpha = \sum_i \sigma_\alpha^i / 2$ as

$$H = -\frac{\lambda}{N} (1 + \gamma) (S^2 - S_z^2 - N/2) - 2hS_z - \frac{\lambda}{2N} (1 - \gamma) (S_+^2 + S_-^2). \quad (2)$$

In the following, for simplicity we set $\lambda = 1$, and since the spectrum of H is even under the transformation $h \leftrightarrow -h$ [23], we restrict our analysis to the region $h \geq 0$. Furthermore, we only consider the maximum spin sector $S = N/2$, to which the ground state is known to belong. A convenient basis of this subspace is spanned by the so-called Dicke states $|N/2, M\rangle$, which are fully symmetric under the permutation group and are eigenstates of S^2 and S_z , with eigenvalues $N(N+2)/4$ and M , respectively.

In order to study the entanglement of the ground state, we need to define an appropriate figure of merit. Following the ideas presented in [3, 4, 10, 11], we consider the von Neumann entropy associated to the ground-state reduced density matrix $\rho_{L,N}$ of a block of size L out of the total N spins, $S_{L,N} = -\text{Tr}(\rho_{L,N} \log_2 \rho_{L,N})$ and analyze its behavior as L is changed, both keeping N finite or sending it to infinity. Note that since the ground state reduced density matrix is spanned by the set of $(L+1)$

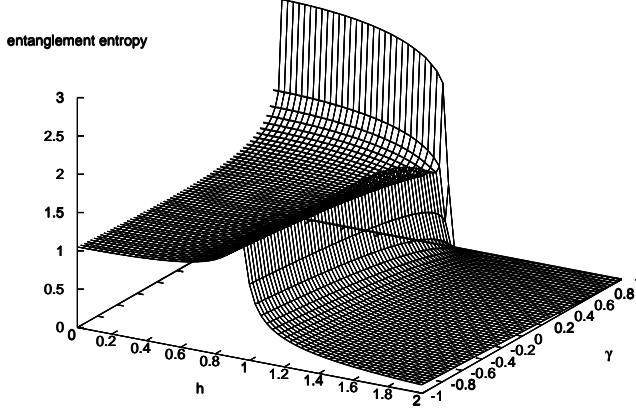


FIG. 1: Entanglement entropy for $N = 500$ and $L = 125$ as a function of h and γ .

Dicke states, the entropy of entanglement obeys the constraint $S_{L,N} \leq \log_2(L+1)$ for all L and N , where the upper bound corresponds to the entropy of the maximally mixed state $\rho_{L,N} = \mathbb{1}/(L+1)$ in the Dicke basis. This argument implies that entanglement, as measured by the Von Neumann entropy, cannot grow faster than the typical logarithmic scaling observed in 1D quantum spin chains [3, 4].

In order to study the different entanglement regimes, we compute the entropy in the plane spanned by γ and h . This numerical computation can be laid out analytically taking advantage of the Hamiltonian symmetries to reduce the complexity of the numerical task to a polynomial growth in N . Results are displayed in Fig. 1 for $N = 500$ and $L = 125$. For $\gamma \neq 1$, one clearly observes an anomaly at the critical point $h = 1$, whereas the entropy goes to zero at large h since the ground state is then a fully polarized state in the field direction. In the zero field limit, the entropy saturates when the size of the system increases and goes to $S_{L,N} = 1$ for $\gamma = 0$, where the ground state approaches a GHZ-like (cat-like) state as in the Ising spin chain [3, 4]. By contrast, for $\gamma = 1$, the entropy increases with the size of the system in the region $0 \leq h < 1$ and jumps to zero at $h = 1$ as we shall now discuss.

In the isotropic case ($\gamma = 1$), it is possible to analytically compute the entropy of entanglement since, at this point, H is diagonal in the Dicke basis. The ground-state energy is given by $E_0(h, \gamma = 1) = -\frac{N}{2} + \frac{2}{N}M^2 - 2hM$, with

$$M = \begin{cases} I(hN/2), & \text{for } 0 \leq h < 1 \\ N/2, & \text{for } h \geq 1 \end{cases}, \quad (3)$$

and the corresponding eigenvector is simply $|N/2, M\rangle$. Here, $I(x)$ denotes the round value of x .

To calculate the entropy, it is convenient to introduce the number n of spins “up” so that $M = n - N/2$, and to write this state in a bipartite form. Indeed, since Dicke states are completely symmetric under any permutation of sites, it is straightforward to see that the ground state can be written as a sum of byproducts of Dicke states:

$$|N/2, n - N/2\rangle = \sum_{l=0}^L p_l^{1/2} |L/2, l - L/2\rangle \otimes |(N-L)/2, n-l - (N-L)/2\rangle, \quad (4)$$

where the partition is made between two blocks of size L and $(N-L)$ and

$$p_l = \frac{\binom{L}{l} \binom{N-L}{n-l}}{\binom{N}{n}}. \quad (5)$$

The ground-state entropy is then simply given by $S_{L,N}(h, \gamma) = -\sum_{l=0}^L p_l \log_2 p_l$. In the limit $N, L \gg 1$, the hypergeometric distribution of the p_l can be recast into a Gaussian distribution $p_l \simeq p_l^g = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(l-\bar{l})^2}{2\sigma^2}\right]$, of mean value $\bar{l} = n\frac{L}{N}$ and variance

$$\sigma^2 = n(N-n)\frac{(N-L)L}{N^3}, \quad (6)$$

where we have retained the subleading term in $(N-L)$ to explicitly preserve the symmetry $S_{L,N} = S_{N-L,N}$. The entropy then reads

$$-\int_{-\infty}^{\infty} dl p_l^g \log_2 p_l^g = \frac{1}{2} (\log_2 e + \log_2 2\pi + \log_2 \sigma^2), \quad (7)$$

and only depends on its variance as expected for a Gaussian distribution. Let us mention that this result has been recently obtained in the context of the ferromagnetic Heisenberg chain [28]. Of course, for $h \geq 1$, the entanglement entropy is exactly zero since the ground state is, in this case, fully polarized in the magnetic field direction ($n = N$). For $h \in [0, 1[$ and in the limit $N, L \gg 1$, Eqs. (3), (6) and (7) lead to

$$S_{L,N}(h, \gamma = 1) \sim \frac{1}{2} \log_2 [L(N-L)/N]. \quad (8)$$

Moreover, the dependence of the entropy with the magnetic field is given by

$$S_{L,N}(h, \gamma = 1) - S_{L,N}(h = 0, \gamma = 1) \sim \frac{1}{2} \log_2 (1 - h^2), \quad (9)$$

and thus diverges, at fixed L and N , in the limit $h \rightarrow 1^-$.

Let us now discuss the more general situation $\gamma \neq 1$, for which no simple analytical solution exists. In this case, the ground state is a superposition of Dicke states with

coefficients that can be easily determined by numerical diagonalizations. Upon tracing out $(N - L)$ spins, each Dicke state decomposes as in Eq. (4). It is then easy to build the $(L + 1) \times (L + 1)$ ground-state reduced density matrix and to compute its associated entropy.

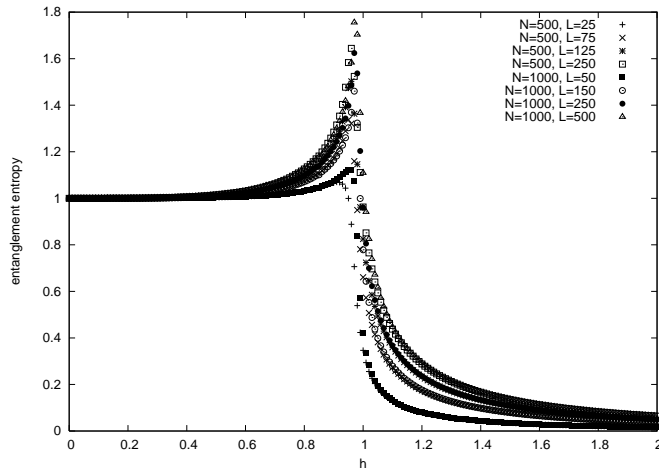


FIG. 2: Entanglement entropy at $\gamma = 0$ as a function of h for different values of N and L . Outside of the critical region, the entropy only depends on the ratio L/N .

We have displayed in Fig. 2, the behavior of the entropy as a function of h , for different values of the ratio L/N and for $\gamma = 0$. For $h \neq 1$, the entropy only depends on the ratio L/N . For any γ , at fixed L/N and in the limit $h \rightarrow \infty$, the entropy goes to zero since the ground state becomes then fully polarized in the field direction. Note that the entropy also vanishes, at $h > 1$, in the limit $L/N \rightarrow 0$ where the entanglement properties are trivial. In the zero field limit, the entropy goes to a constant which depends on γ and equals 1 at $\gamma = 0$ since the ground state is then a GHZ-like state made up of spins pointing in $\pm x$ directions. Close to criticality, the entropy displays a logarithmic divergence, which we numerically found to obey the law

$$S_{L,N}(h, \gamma) \sim -a \log_2 |1 - h|, \quad (10)$$

where a is very close to $1/6$ for $N, L \gg 1$ as can be seen in Fig. 3. At the critical point, the entropy has a nontrivial behavior that we have studied focusing on the point $\gamma = 0$ which is representative of the class $\gamma \neq 1$. There, the entropy also scales logarithmically with L as in the isotropic case, but with a different prefactor. More precisely, we find

$$S_{L,N}(h = 1, \gamma \neq 1) \sim b \log_2 [L(N - L)/N]. \quad (11)$$

For the finite-size systems investigated here, the prefactor varies when either the ratio L/N or γ is changed, as can be seen in Fig. 4. However, in the thermodynamical limit $N, L \gg 1$ (and finite L/N), it is likely that $b = 1/3$ and does not depend on γ . In addition, at fixed L and

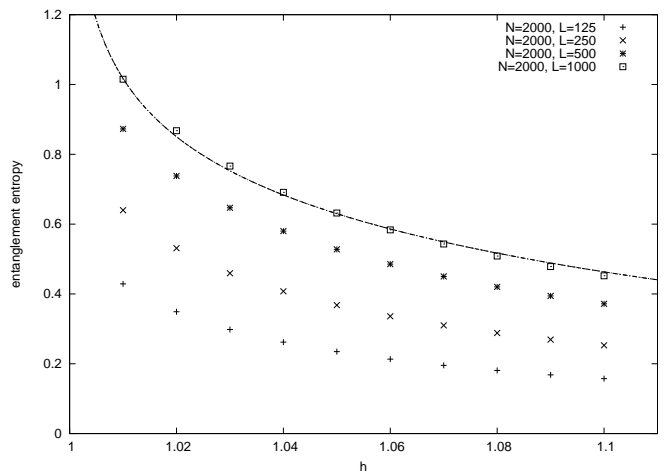


FIG. 3: Entanglement entropy as a function of h near the critical point for $\gamma = 0$. The solid line corresponds to the fitting law (10) with $a = 1/6$.

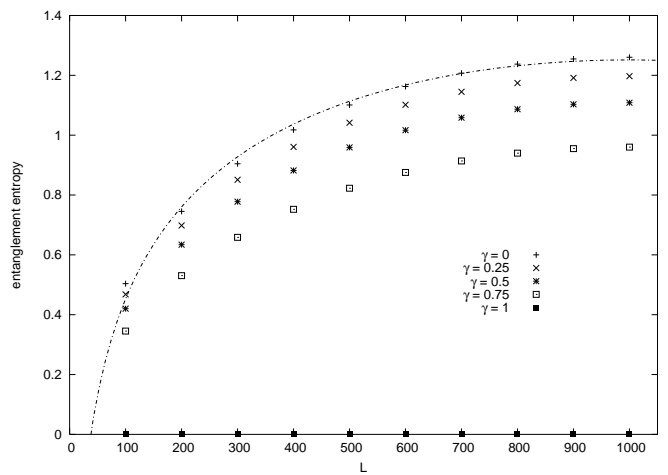


FIG. 4: Entanglement entropy as a function of L at the critical point for different γ and $N = 2000$. The solid line corresponds to the fitting law (11) with $b = 1/3$.

N , the entropy also depends on the anisotropy parameter logarithmically as

$$S_{L,N}(h = 1, \gamma) - S_{L,N}(h = 1, \gamma = 0) \sim d \log_2 (1 - \gamma), \quad (12)$$

for all $-1 \leq \gamma < 1$, as can be seen in Fig. 5. Here again, it is likely that, in the thermodynamical limit, d has a simple (rational) value which, from our data, seems to be $1/6$. It is important to keep in mind that the limit $\gamma \rightarrow 1$ and the thermodynamical limit do not commute, so that Eq. (12) is only valid for $\gamma \neq 1$.

Let us now compare these results with those found in the 1D XY model. As for the LMG model, the XY chain has two different universality classes depending on the anisotropy parameter. At the critical point, the entropy

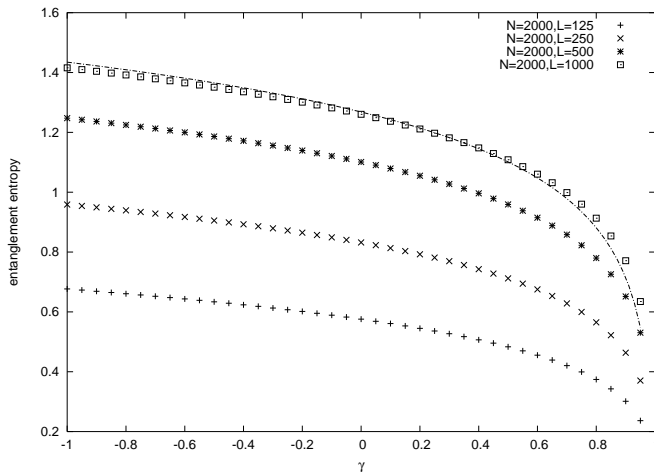


FIG. 5: Entanglement entropy at the critical point $h = 1$ as a function of γ . The solid line corresponds to the fitting law (12) with $d = 1/6$.

has been found to behave as [3, 4, 29]

$$S_{L,N} \sim \frac{c}{3} \log_2[L(N-L)/N], \quad (13)$$

where c is the central charge of the corresponding conformal theory [30]. For the isotropic case, the critical model is indeed described by a free boson theory with $c_{XX} = 1$ whereas the anisotropic case corresponds to a free fermion theory with $c_{XY} = 1/2$. It is striking to see that the entropy in the LMG model has the same logarithmic dependence with some prefactor which, as in the 1D case, depends only on the universality classes [see Eqs. (8) and (11)]. Concerning the dependence with the magnetic field and with the anisotropy parameter, it is also worth noting that logarithmic behaviors (9), (10), and (12) are similar to those found in the XX and XY chain [31] except that the prefactor in the LMG model are different. On top, the behavior of this model with respect to majorization for $\gamma \neq 1$ and as h departs from its critical value is completely analogous to the 1D XY model [32], namely, all the eigenvalues of the reduced density matrix obey a strict majorization relation as h grows, while for decreasing h one of these eigenvalues drives the system towards a GHZ-like state in such a way that majorization is strictly obeyed in the thermodynamical limit. This behavior implies a very strong “sense of order” in the ground state, in complete analogy to the XY chain.

The logarithmic scaling with L of the ground-state entanglement entropy we have found for the LMG model, with well-defined values of the scaling coefficients in the whole parameter space, entices the search for a precise construction of an underlying one-dimensional local field theory. In order to further clarify this statement, we have explicitly analyzed the distribution and degeneracy of eigenvalues of the reduced density matrix of the sys-

tem, that is, the structure of the partition function of the model. We have observed that, at $\gamma = 1$, the spectrum is doubly-degenerate with the exception of the first eigenvalue and follows a Gaussian behavior (as expected from the previous analytical calculations), while at $\gamma = 0$ the spectrum is doubly degenerate and equally spaced in logarithmic scale. The behavior of the spectrum for $\gamma = 0$ accommodates to the typical structure imposed by the Virasoro algebra over the highest-weight operators in conformal field theories [33]. This observation might be a hint of a possible conformal invariance underlying the LMG model in some regions of its parameter space.

Finally, let us mention that during the completion of this work, the entanglement entropy has also been calculated for the antiferromagnetic LMG model [34] for which the ground state is known exactly [20, 35].

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